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Electromagnetic Field in Finsler and Associated Spaces

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The electromagnetic field and its interaction with the leptons is introduced in Finsler space. This space is also considered as the microlocal space-time of the extended hadrons. The field equations for the Finsler space have been obtained from the classical field equations by quantum generalization of this space-time below a fundamental length-scale. On the other hand, the classical field equations are derived from a property of the fields on the autoparallel curve of the Finsler space. The field equations for the associated spaces of the Finsler space, which are macroscopic spaces, such as the large-scale space-time of the universe and the usual Minkowski space-time, can also be obtained for the case of Finslerian bispinor fields separable as the direct products of fields depending on the position coordinates with those depending on the directional arguments. The equations for the coordinate-dependent fields are the usual field equations with the cosmic time-dependent masses of the leptons. The other equations of the directional variable-dependent fields are solved here. Also, the lepton current and the continuity equation are considered. The form-invariance of the field equations under the general coordinate transformations of the Finsler spaces has been discussed.

KEY WORDS: bispinor; microdomain; macrospace; extended hadron; directional variable.

1. INTRODUCTION

Recently, in De (1991, 1997) the microlocal space-time of extended hadrons has been considered as an anisotropic space-time. This space-time is regarded as a special type of Finsler space. In fact, the extended hadron-structure in such an anisotropic microscopic domain corresponds to the picture of composite character of hadrons, the origin of which lies in the works of Sakata (1956), Gell-Mann (1964), Bjorken (1969), and Feynman (1972). Also, Yukawa (1948, 1950) opened the possibility of intrinsic extensions of the subatomic particles in the microscopic space-time. One is motivated to consider such a theory for understanding multifariousness of elementary particles (whose number went on increasing in last 50 years) as the ultimate manifestation of their extensions in the microscopic space-time

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which may be different from the macroscopic one. In a previous article (De, 1997) it was, indeed, possible to find the geometrical origin for both the internal quantum numbers of the constituents of hadrons and the internal symmetry of hadrons.

Another aspect regarding the microscopic space-time is that below a fundamental length-scale the space-time should be regarded as quantum one. In fact, below the Planck length-scale the space-time may not be a meaningful concept in the classical sense. This is evident from the uncertainty principle in Adler and Santiago (1999) who have modified the usual uncertainty relation by considering the gravitational interaction of the photon and the particle. This is also a standard result of the superstring theory. The intrinsic limitation to quantum measurements of space-time distances has also been discussed by Ng and Van Dam (1994; and references therein). This aspect of quantum generalization of microscopic space-time has been taken into account in deriving the field equations for the quantum fields in that space (De, 1997, 2001). These fields and their equations are obtained in order to apprehend the interactions of hadrons. There we have found the field equations for a free lepton as well as lepton interacting with the external electromagnetic field both for the Finslerian microscopic space and the macrospaces which appear as the associated curved (Riemannian) or flat spaces of the Finsler space.

In the present article, the field equations for the interacting lepton and electromagnetic fields in the Finsler space will be found. For the case of the fields of this space, separable as the direct products of fields, one depending only on the coordinates and the other on the directional arguments of the Finsler space, the former one corresponds to the field of the associated space of the Finsler space. The field equations for these fields have been shown to be the usual field equations of the macroscopic spaces which may be the large-scale space-time of the universe or the laboratory space-time (that is, the Minkowski space-time). Of course, there appears an additional cosmological time-dependent mass of the particle, which has no appreciable contribution in the present era of the universe. But it was dominant in the very early period of the universe and has cosmological consequences (De, 1993, 1999). The fields dependent on the directional variables are found to satisfy different field equations which are solved here. Also, the lepton current in the Finsler space has been obtained. All these considerations are intended in order to develop the field theory in this space.

This article is organized as follows. In Section 2, the electromagnetic field has been introduced for the Finsler space and by quantum generalization of this space-time the field equations are obtained. These fields are found to satisfy Bargmann–Wigner type equations. The Maxwell's equations for the Minkowski space-time have also been derived here. In Section 3, the electromagnetic interaction with the spinor (lepton) fields has been considered. The field equations are obtained from a property of the classical fields on the neighboring points of the autoparallel curve of the Finsler space and by quantum generalization of that space-time below a fundamental length-scale. The Finslerian microlocal space-time of extended hadrons is specified here in accord with the Riemann's original suggestion (Riemann, 1854). The separations of the fields are also been made here, and for the separated fields the corresponding field equations are found. The epoch-dependent additional masses appear in these field equations. In Section 4, the directional variable-dependent fields have been found as the solutions of their equations. The lepton current in the Finsler space and the continuity equation are considered here. Section 5 concludes this article with a brief summary of the results and a discussion on the form-invariances of the field equations in the Finsler spaces.

2. ELECTROMAGNETIC FIELD IN FINSLER SPACE

Here, we introduce the electromagnetic field (the photon field) in the Finsler space. Firstly, the "classical" field $\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$ is presented. This field satisfies an "equivalence property" for each pair of indices (i', j'). This property of the field is to be satisfied by it on the autoparallel curve of the Finsler space (De, 1997). For a free field this is given as (on the autoparallel curve)

$$\begin{aligned} \delta\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) &= dx^{\mu}\partial_{\mu}\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) \\ &\quad + d\nu^{\ell}\partial'_{\ell}\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) = 0 \quad \text{for each pair } (i', j'), \end{aligned} \tag{1}$$

where,

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} \quad \text{and} \quad \partial'_{\ell} \equiv \frac{\partial}{\partial \nu^{\ell}}$$

The differentials dx^{μ} and $d\nu^{\ell}$ are related by

$$d\nu^{\ell} = -\gamma^{\ell}_{hj}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^{h}dx^{j} \tag{2}$$

where $\gamma^{\ell}_{hj}(\underset{\sim}{x}, \underset{\sim}{\nu})$ are the Christoffel symbols of second kind.

Now, we make the quantum generalization of the Finslerian microscopic space-time below a fundamental length-scale in two steps (De, 1997, 2001), namely,

$$\text{(i) } dx^{\mu} \to \Delta\hat{x}^{\mu} \quad \text{and} \quad \text{(ii) } d\nu^{\ell} \to \Delta\hat{\nu}^{\ell}$$

where $\Delta\hat{x}^{\mu} = \epsilon\sqrt{\theta(\underset{\sim}{\nu}^2)}\gamma^{\mu}(\underset{\sim}{x})$. Here, $\gamma^{\mu}(\underset{\sim}{x})$ $(\mu = 0, 1, 2, 3)$ are Dirac matrices for the associated curved space (Riemannian) of the Finsler space, and $\underset{\sim}{\nu}^2 = \eta_{\mu\nu}\nu^{\mu}\nu^{\nu}$.

The function θ is given by

$$\begin{aligned} \theta(z) &= 1 \qquad \text{for } z \geq 0 \\ &= -1 \quad \text{for } z < 0 \end{aligned} \tag{3}$$

With this quantization one can arrive at the following equation for the quantum field $\psi_{(i,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$, which becomes a "bispinor" for each (i', j'):

$$\gamma^{\mu}(\underset{\sim}{x})\big(\partial_{\mu} - \gamma^{\ell}_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^{h}\partial'_{\ell}\big)\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) = 0 \tag{4}$$

As $\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$ is now a bispinor for each (i', j'), the above equation is, in fact, a compact form of the following equation:

$$\gamma^{\mu}_{ik}(\underset{\sim}{x})\partial_\mu \psi_{(k,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) - \gamma^{\mu}_{jk}(\underset{\sim}{x})\gamma^{\ell}_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^h \partial'_\ell \psi_{(i,k),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) = 0 \quad \text{for each } (i', j'). \tag{5}$$

Similarly, one can consider the classical field $\psi_{(i,j)}(\underset{\sim}{x}, \underset{\sim}{\nu})$ and proceeds, as above, with its property on the autoparallel curve and the space-time quantization of the Finslerian microdomain to arrive at the following equation for the other pair of indices (i', j'):

$$\gamma^{\mu}_{i'k}(\underset{\sim}{x})\partial_\mu \psi_{(i,j),(k,j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) - \gamma^{\mu}_{j'k}(\underset{\sim}{x})\gamma^{\ell}_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^h \partial'_\ell \psi_{(i,j),(i',k)}(\underset{\sim}{x}, \underset{\sim}{\nu}) = 0 \quad \text{for each } (i, j). \tag{6}$$

Thus, $\psi_{(i,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$ is completely symmetric with respect to the pairs (i, j) and (i', j'). These equations are analogous to Bargmann–Wigner equations for spin $s \geq 1/2$ particles (Bargmann and Wigner, 1948; Dirac, 1936), where, of course, the equations (Dirac equations for each index) have to be postulated.

Now, when the field $\psi_{(i,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$ is separable as the direct product of two bispinors $\hat{\psi}_{ii'}(\underset{\sim}{x})$ and $\hat{\phi}_{jj'}(\underset{\sim}{\nu})$ (represented as 4×4 matrices), one depending on the coordinates x^μ and the other on the directional variables ν^μ of the Finsler space, that is,

$$\left\{\psi_{(i,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})\right\} = \{\hat{\psi}_{ii'}(\underset{\sim}{x})\} \times \{\hat{\phi}_{jj'}(\underset{\sim}{\nu})\} \tag{7}$$

where $\hat{\psi}(\underset{\sim}{x}) = \{\hat{\psi}_{ii'}(\underset{\sim}{x})\}$ and $\hat{\phi}(\underset{\sim}{\nu}) = \{\hat{\phi}_{jj'}(\underset{\sim}{\nu})\}$ are completely symmetric with respect to their respective indices, then we have from (5)

$$\gamma^{\mu}_{ik}(\underset{\sim}{x})\partial_\mu \hat{\psi}_{ki'}(\underset{\sim}{x}) \times \hat{\phi}_{jj'}(\underset{\sim}{\nu}) - \hat{\psi}_{ii'}(\underset{\sim}{x}) \times \gamma^{\mu}_{jk}(\underset{\sim}{x})\gamma^{\ell}_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^h \partial'_\ell \hat{\phi}_{kj'}(\underset{\sim}{\nu}) = 0 \quad \text{for each } (i', j') \tag{8}$$

It should be noted that the Dirac matrices $\gamma^\mu(\underset{\sim}{x})$ for the associated curved space of the Finsler space are connected with the flat space Dirac matrices γ^α through the vierbeins $V^\mu_\alpha(\underset{\sim}{x})$, that is,

$$\gamma^\mu(\underset{\sim}{x}) = V^\mu_\alpha(\underset{\sim}{x})\gamma^\alpha \tag{9}$$

We shall later consider the Finslerian microscopic space-time for which the coefficients $\gamma^{\ell}_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})$ are separable into two functions, one depending on the coordinates x^μ and the other on the directional arguments ν^μ. Now, if the bispinor $\hat{\phi}(\underset{\sim}{\nu})$ satisfies the following equation

$$\gamma^{\mu}_{jk}(\underset{\sim}{x})\gamma^{\ell}_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^h \partial'_\ell \hat{\phi}_{kj'}(\underset{\sim}{\nu}) = 0 \tag{10}$$

then we have from (8),

$$\gamma^{\mu}_{ik}(\underset{\sim}{x})\partial_\mu \hat{\psi}_{ki'}(\underset{\sim}{x}) = 0, \quad \text{for each } i'. \tag{11}$$

We shall see later that Eq. (10) will be independent of the coordinates x^μ for the case of the Finsler space of extended hadrons. Again, for the flat space (Minkowskian space-time), the Eq. (11) becomes

$$\gamma^{\alpha}_{ik}\partial_\alpha \hat{\psi}_{ki'}(\underset{\sim}{x}) = 0, \quad \text{for each } i' \tag{12}$$

Similarly, from (6) we can arrive at the equation

$$\gamma^{\alpha}_{i'k}\partial_\alpha \hat{\psi}_{ik}(\underset{\sim}{x}) = 0, \quad \text{for each } i \tag{13}$$

Equations (12) and (13) are the usual Bargmann–Wigner equations for a massless spin 1 particle, that is, for photon. These equations can also be written as

$$\begin{aligned} \gamma \cdot \partial \hat{\psi}(\underset{\sim}{x}) &= 0 \\ \hat{\psi}(\underset{\sim}{x})\gamma^T \cdot \overleftarrow{\partial} &= 0 \end{aligned} \tag{14}$$

Since $\hat{\psi}(\underset{\sim}{x})$ is symmetric (that is, $\hat{\psi}^T(\underset{\sim}{x}) = \hat{\psi}(\underset{\sim}{x})$), it can be expressible as (Bargmann and Wigner, 1948; Lurié, 1968)

$$\hat{\psi}(\underset{\sim}{x}) = \frac{1}{2}\sigma^{ab}CF_{ab}(\underset{\sim}{x}) \tag{15}$$

where $\sigma^{ab} = \frac{i}{2}(\gamma^a\gamma^b - \gamma^b\gamma^\alpha)$, and C is the charge conjugation matrix given by $C = i\gamma^2\gamma^o$. From (14) and (15), it is easy to see that $F_{ab}(\underset{\sim}{x})$ satisfies the following equation:

$$[\gamma^\alpha, \sigma^{ab}]C\partial_\alpha F_{ab}(\underset{\sim}{x}) = 0 \tag{16}$$

In deducing it, the relation $C^{-1}\gamma^\mu C = -\gamma^{\mu^T}$ has been used. Since, $[\gamma^\alpha, \sigma^{ab}] = 2i(g^{\alpha a}\gamma^b - g^{\alpha b}\gamma^a)$ where $g^{ab} = \text{diag}(1, -1, -1, -1)$, we obtain

$$\gamma^a C\partial^b F_{ab}(\underset{\sim}{x}) = 0$$

which leads to the Maxwell's equation

$$\partial^b F_{ab}(\underset{\sim}{x}) = 0 \tag{17}$$

In terms of the electromagnetic potential field $A_\mu(\underset{\sim}{x})$, we have

$$F_{ab}(\underset{\sim}{x}) = \partial_a A_b(\underset{\sim}{x}) - \partial_b A_a(\underset{\sim}{x}), \tag{18}$$

and with the Lorentz condition

$$\partial^a A_a(\underset{\sim}{x}) = 0 \tag{19}$$

we have from (17),

$$\Box A_a(\underset{\sim}{x}) = 0 \quad \text{where} \quad \Box = \partial^\alpha\partial_\alpha \tag{20}$$

Thus, the "$\underset{\sim}{x}$-part" of the field of the Finsler space appears as the usual electromagnetic field which satisfies the Maxwell's equations in the Minkowski space-time. Of course, the Finslerian electromagnetic fields satisfy Bargmann–Wigner type equations (5) and (6).

3. ELECTROMAGNETIC INTERACTION

Now, we introduce the electromagnetic interaction with the spinor (lepton) fields. For this purpose, we use the properties of the classical fields on the autoparallel curve of the Finsler space, and then by quantum generalization of this space-time we arrive at the quantum fields and their equations. These properties of the classical fields on the autoparallel curve are expressed by the following equations:

$$\delta(\psi\chi) + \frac{ie}{\hbar c}(A_\mu\, dx^\mu)\psi\chi = -\frac{i\epsilon mc}{\hbar}\sqrt{\theta(\underset{\sim}{\nu}^2)}\psi\chi, \tag{21}$$

and

$$\delta\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) = -i\epsilon\, e\sqrt{\theta(\underset{\sim}{\nu}^2)}\tilde{C}_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) \quad \text{for each } (i', j') \tag{22}$$

Here, $\chi(\underset{\sim}{x}, \underset{\sim}{\nu})$ is a scalar function which may be regarded as a phase factor for the lepton field $\psi(\underset{\sim}{x}, \underset{\sim}{\nu})$. The vector field $A_\mu(\underset{\sim}{x}, \underset{\sim}{\nu})$ represents the electromagnetic field. The field $\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$ gives rise to the electromagnetic field also through quantum generalization of the Finslerian microscopic space-time. $\tilde{C}_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$, for each (i', j'), is proportional to the change in the field $\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$ on the autoparallel curve and becomes a bispinor for each (i', j') on quantization. This field is, in fact, connected with the interacting spinor (lepton) field, and characterization of it will be made later. Here, the mass term m and e appearing in the constants of proportionality are the "inherent" mass of the particle and the electric charge respectively. ϵ is a real parameter such that $0 < \epsilon \leq \ell$, ℓ being a fundamental length. Equation (21) is the same as that for the case of interaction with an external electromagnetic field considered earlier (De, 2001). Therefore we get the same equation for the quantum bispinor field $\psi_{\alpha\beta}(\underset{\sim}{x}, \underset{\sim}{\nu})$ of the lepton interacting with the electromagnetic field which is separable as

$$A_\mu(\underset{\sim}{x}, \underset{\sim}{\nu}) = A_\mu(\underset{\sim}{x}) - \gamma^\ell_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^h \tilde{A}_\ell(\underset{\sim}{\nu}) \tag{23}$$

(As pointed out earlier that the coefficients $\gamma^\ell_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})$ can also be made separable for the Finslerian microscopic space-time). This equation is given as

$$\gamma^\mu_{\alpha\alpha'}(\underset{\sim}{x})\left(i\hbar\partial_\mu - \frac{e}{c}\bar{A}_\mu(\underset{\sim}{x})\right)\psi_{\alpha'\beta}(\underset{\sim}{x}, \underset{\sim}{\nu}) - i\hbar\gamma^\mu_{\beta\beta'}(\underset{\sim}{x})\gamma^\ell_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu}) \times \nu^h\partial'_\ell\psi_{\alpha\beta'}(\underset{\sim}{x}, \underset{\sim}{\nu}) = mc\psi_{\alpha\beta}(\underset{\sim}{x}, \underset{\sim}{\nu}) \tag{24}$$

where

$$\bar{A}_\mu(\underset{\sim}{x}) = A_\mu(\underset{\sim}{x}) + \frac{1}{e}\partial_\mu \phi(\underset{\sim}{x}) \tag{25}$$

which is the usual gauge transformation in the associated curve space (Riemannian) of the Finsler space.

On the other hand, we can proceed as before for Eq. (22) which expresses the change of the field $\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$ on the autoparallel curve. That is, we have on the autoparallel curve

$$\begin{aligned} &dx^\mu \partial_\mu \psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) + d\nu^\ell \partial'_\ell \psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) \\ &\quad = -i\epsilon\, e\sqrt{\theta(\nu^2)}\tilde{C}_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) \quad \text{for each } (i', j') \end{aligned} \tag{26}$$

When we make the quantum generalization of the Finsler space in two steps, namely,

$$\text{(i)} \quad dx^\mu \to \Delta\hat{x}^\mu = \epsilon\sqrt{\theta(\underset{\sim}{\nu}^2)}\gamma^\mu(\underset{\sim}{x}), \tag{27a}$$

$$\text{(ii)} \quad d\nu^\ell \to \Delta\hat{\nu}^\ell = -\epsilon\sqrt{\theta(\underset{\sim}{\nu}^2)}\gamma^\ell_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^h\gamma^\mu(\underset{\sim}{x}), \tag{27b}$$

the field $\psi_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$ for each (i', j') becomes a bispinor $\psi_{(i,j)(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$. Also, due to this transition we have

$$\begin{aligned} &\tilde{C}_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) \to \tilde{C}_{(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})(\text{a bispinor or a } 4 \times 4 \text{ matrix for each } (i', j')) \\ &\quad = \left\{\tilde{C}_{(i,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})\right\} \end{aligned}$$

Consequently, Eq. (26) changes to

$$\begin{aligned} &\gamma^\mu_{ik}(\underset{\sim}{x})\partial_\mu \psi_{(k,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) - \gamma^\mu_{jk}(\underset{\sim}{x})\gamma^\ell_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^h \partial'_\ell \psi_{(i,k)(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) \\ &\quad = -ie\, \tilde{C}_{(i,j)(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) \quad \text{for each } (i', j') \end{aligned} \tag{28}$$

Similarly, starting from the classical field $\psi_{(i,j)}(\underset{\sim}{x}, \underset{\sim}{\nu})$ one can arrive at the following equation for each (i, j):

$$\begin{aligned} &\gamma^\mu_{i'k}(\underset{\sim}{x})\partial_\mu \psi_{(i,j)(k,j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) - \gamma^\mu_{j'k}(\underset{\sim}{x})\gamma^\ell_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^h \partial'_\ell \psi_{(i,j)(i',k)}(\underset{\sim}{x}, \underset{\sim}{\nu}) \\ &\quad = -\, ie\, \tilde{C}_{(i,j)(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu}) \quad \text{for each } (i, j) \end{aligned} \tag{29}$$

Both $\psi_{(i,j)(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$ and $\tilde{C}_{(i,j)(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})$ are completely symmetric with respect to the pairs of indices (i, j) and (i', j') as in the free-field case considered in the previous section. The set of equations (24), (28), and (29) are the field equations for the interacting fields in the Finsler space.

Now, the specification of $\tilde{C}(\underset{\sim}{x}, \underset{\sim}{\nu}) = \{\tilde{C}_{(i,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})\}$ can be made by the direct products of the fields represented as matrices. That is, for the case of separable field $\{\psi_{(i,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})\}$ as

$$\left\{\psi_{(i,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})\right\} = \{\hat{\psi}_{ii'}(\underset{\sim}{x})\} \times \{\hat{\phi}_{jj'}(\underset{\sim}{\nu})\}, \tag{30}$$

$\tilde{C}(\underset{\sim}{x}, \underset{\sim}{\nu})$ can be specified as

$$\left\{\tilde{C}_{(i,j),(i',j')}(\underset{\sim}{x}, \underset{\sim}{\nu})\right\} = \{\xi_{ii'}(\underset{\sim}{x})\} \times \{\hat{\phi}_{jj'}(\underset{\sim}{\nu})\} \tag{31}$$

where

$$\hat{\psi}(\underset{\sim}{x}) = \{\hat{\psi}_{ii'}(\underset{\sim}{x})\}, \quad \hat{\phi}(\underset{\sim}{\nu}) = \{\hat{\phi}_{jj'}(\underset{\sim}{\nu})\}, \quad \text{and} \quad \xi(\underset{\sim}{x}) = \{\xi_{ii'}(\underset{\sim}{x})\}$$

are completely symmetric with respect to their indices. $\xi(\underset{\sim}{x})$ is yet to be specified. With these separations, we have, from (28),

$$\begin{aligned}\left\{\gamma^{\mu}_{ik}(\underset{\sim}{x})\partial_{\mu}\hat{\psi}_{ki'}(\underset{\sim}{x})\right\} \times \{\hat{\phi}_{jj'}(\underset{\sim}{\nu})\} - \{\hat{\psi}_{ii'}(\underset{\sim}{x})\} \times \left\{\gamma^{\mu}_{jk}(\underset{\sim}{x})\gamma^{\ell}_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^{h}\partial'_{\ell}\hat{\phi}_{kj'}(\underset{\sim}{\nu})\right\} \\ = \{-ie\,\xi_{ii'}(\underset{\sim}{x})\} \times \{\hat{\phi}_{jj'}(\underset{\sim}{\nu})\} \quad \text{for each } (i', j')\end{aligned} \tag{32}$$

Similarly, from (29) we have

$$\begin{aligned}\left\{\gamma^{\mu}_{i'k}(\underset{\sim}{x})\partial_{\mu}\hat{\psi}_{ik}(\underset{\sim}{x})\right\} \times \{\hat{\phi}_{jj'}(\underset{\sim}{\nu})\} - \{\hat{\psi}_{ii'}(\underset{\sim}{x})\} \times \left\{\gamma^{\mu}_{j'k}(\underset{\sim}{x})\gamma^{\ell}_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^{h}\partial'_{\ell}\hat{\phi}_{jk}(\underset{\sim}{\nu})\right\} \\ = \{-ie\,\xi_{ii'}(\underset{\sim}{x})\} \times \{\hat{\phi}_{jj'}(\underset{\sim}{\nu})\} \quad \text{for each } (i, j)\end{aligned} \tag{33}$$

Now, as before, if $\hat{\phi}(\underset{\sim}{\nu})$ satisfies the following equations:

$$\begin{aligned}\gamma^{u}_{jk}(\underset{\sim}{x})\gamma^{\ell}_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^{h}\partial'_{\ell}\hat{\phi}_{kj'}(\underset{\sim}{\nu}) = 0, \\ \gamma^{u}_{j'k}(\underset{\sim}{x})\gamma^{\ell}_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^{h}\partial'_{\ell}\hat{\phi}_{jk}(\underset{\sim}{\nu}) = 0,\end{aligned} \tag{34}$$

then we have from (32) and (33) the following equations for $\hat{\psi}(\underset{\sim}{x})$:

$$\begin{aligned}\gamma^{\mu}_{ik}(\underset{\sim}{x})\partial_{\mu}\hat{\psi}_{ki'}(\underset{\sim}{x}) + ie\,\xi_{ii'}(\underset{\sim}{x}) = 0 \\ \gamma^{\mu}_{i'k}(\underset{\sim}{x})\partial_{\mu}\hat{\psi}_{ik}(\underset{\sim}{x}) + ie\,\xi_{ii'}(\underset{\sim}{x}) = 0\end{aligned} \tag{35}$$

In De (1997) the microlocal space-time of extended hadrons has been specified as a special Finsler space, the fundamental function of which is given by

$$F^{2}(\underset{\sim}{x}, \underset{\sim}{\nu}) = \hat{g}_{ij}(\underset{\sim}{x}, \underset{\sim}{\nu})\nu^{i}\nu^{j} \tag{36}$$

where

$$\hat{g}_{ij}(\underset{\sim}{x}, \underset{\sim}{\nu}) = \eta_{ij}\,g(\underset{\sim}{x})\theta(\underset{\sim}{\nu}^{2}),$$

and

$$\eta_{ij} = \text{diag.}(+1, -1, -1, -1).$$

Here, $\hat{g}$ is not, in general, the Finsler metric tensor, but simply represents a homogeneous tensor of degree zero in $\underset{\sim}{\nu}$, which is used for the purpose of defining F.

The Finsler metric can, in fact, be obtained with the use of the following formula:

$$g_{ij}(\underset{\sim}{x}, \underset{\sim}{\nu}) = \frac{1}{2}\frac{\partial^2 F^2(\underset{\sim}{x}, \underset{\sim}{\nu})}{\partial \nu^i \partial \nu^j} \tag{37}$$

The Finsler space introduced here is in accord with Riemann's original suggestion (Riemann, 1854) that the positive fourth root of a fourth order differential form might serve as a metric function. Actually, from a fundamental function given by

$$F(\underset{\sim}{x}, \underset{\sim}{\nu}) = \{g_{\mu\nu\rho\sigma}(\underset{\sim}{x})\nu^\mu \nu^\nu \nu^\rho \nu^\sigma\}^{1/4} \tag{38}$$

with the tensor field

$$g = (g_{\mu\nu\rho\sigma}(\underset{\sim}{x})) = (\{g(\underset{\sim}{x})\}^2 \eta_{\mu\nu}\eta_{\rho\sigma}), \tag{39}$$

one can have (De, 2001)

$$ds^4 = \{g(\underset{\sim}{x})\eta_{\mu\nu}\, dx^\mu\, dx^\nu\}^2. \tag{40}$$

This metric relation (40) has, in fact, two possibilities. The first one corresponds to a Riemannian space, the metric of which is

$$ds^2 = g(\underset{\sim}{x})\eta_{\mu\nu}\, dx^\mu\, dx^\nu,$$

and the other one gives rise to the fundamental function (36) of a Finsler space. It is to be noted that if one insists on the condition of nonnegativeness of ds^2, then the second possibility from (40) conforms this requirement, and we regard this Finsler space as the microlocal space-time of hadronic matter-extension. The present Finsler space $F_4 = (M_4, F)$ is a simple type of (α, β)-metric Finsler space. The fundamental function F of this space can be written as $F(\underset{\sim}{x}, \underset{\sim}{\nu}) = \alpha\sqrt{\theta(\alpha^2)} = \alpha\beta^o\sqrt{\theta(\alpha^2)}$ with $\alpha(\underset{\sim}{x}, \underset{\sim}{\nu}) = (g(\underset{\sim}{x})\underset{\sim}{\nu}^2)^{1/2}$, $g(\underset{\sim}{x}) > 0$, and $\beta(\underset{\sim}{x}, \underset{\sim}{\nu})$ is a differential 1-form. Obviously, the function F is (1) p-homogeneous in α and β. The associated space $R_4 = (M_4, \alpha)$ is Riemannian. This space is conformal to the Minkowski space-time. The matrices $\gamma^\mu(\underset{\sim}{x})$ of this associated curved space-time are related to the flat space (Dirac) matrices through the vierbein $V^\mu_\alpha(\underset{\sim}{x})$ as follows:

$$\gamma^\mu(\underset{\sim}{x}) = V^\mu_\alpha(\underset{\sim}{x})\gamma^\alpha \tag{41}$$

For the present case, the vierbein fields $V^\mu_\alpha(\underset{\sim}{x})$ and $V^\alpha_\mu(\underset{\sim}{x})$ are given as

$$V^\mu_\alpha(\underset{\sim}{x}) = \{g(\underset{\sim}{x})\}^{-1/2}\delta^\mu_\alpha, \qquad V^\alpha_\mu(\underset{\sim}{x}) = \{g(\underset{\sim}{x})\}^{1/2}\delta^\alpha_\mu \qquad V^\mu_\alpha(\underset{\sim}{x})V^\beta_\mu(\underset{\sim}{x}) = \delta^\beta_\alpha \tag{42}$$

With these vierbein fields the above Eqs. (35) become for the flat space (that is, for Minkowski space-time)

$$\begin{aligned} (\gamma^\alpha)_{ik}\partial_\alpha \hat{\psi}_{ki'}(\underset{\sim}{x}) + ie\,\xi_{ii'}(\underset{\sim}{x}) &= 0 \\ (\gamma^\alpha)_{i'k}\partial_\alpha \hat{\psi}_{ik}(\underset{\sim}{x}) + ie\,\xi_{ii'}(\underset{\sim}{x}) &= 0 \end{aligned} \tag{43}$$

Now, the symmetric bispinor $\hat{\psi}(\underset{\sim}{x})$ (represented as a 4×4 matrix) is expressible as in (15), and $\xi(\underset{\sim}{x})$ which is also symmetric can now be specified as

$$\xi(\underset{\sim}{x}) = \gamma^{\alpha} C(\bar{\psi}(\underset{\sim}{x})\gamma_{\alpha}\psi(\underset{\sim}{x})) \tag{44}$$

where C is the charge conjugation matrix. $\gamma^{\alpha} C$ is also symmetric. Here, $\psi(\underset{\sim}{x})$ and $\bar{\psi}(\underset{\sim}{x})$ are, respectively, the "$\underset{\sim}{x}$-parts" of the separated (lepton) field $\psi(\underset{\sim}{x}, \underset{\sim}{\nu})$ and its adjoint field $\bar{\psi}(\underset{\sim}{x}, \underset{\sim}{\nu})$ which is defined as

$$\bar{\psi}(\underset{\sim}{x}, \underset{\sim}{\nu}) = \gamma^{o}\psi(\underset{\sim}{x}, \underset{\sim}{\nu})^{\dagger}\gamma^{o}, \quad \text{and therefore} \quad \psi(\underset{\sim}{x}, \underset{\sim}{\nu}) = \gamma^{o}\bar{\psi}(\underset{\sim}{x}, \underset{\sim}{\nu})^{\dagger}\gamma^{o} \tag{45}$$

This definition of adjoint field intended for the bispinor field in the Finsler space of our consideration. In fact, the field $\psi(\underset{\sim}{x}, \underset{\sim}{\nu})$ (represented as a 4×4 matrix) is separated as

$$\psi(\underset{\sim}{x}, \underset{\sim}{\nu}) = \psi(\underset{\sim}{x}) \times \phi(\underset{\sim}{\nu})^{T} = \phi(\underset{\sim}{\nu})^{T} \times \psi(\underset{\sim}{x}) = \psi(\underset{\sim}{x})\,\phi(\underset{\sim}{\nu})^{T} \tag{46}$$

where $\psi(\underset{\sim}{x})$ and $\phi(\underset{\sim}{\nu})$ are spinors and are represented as 4×1 matrices. Thus, the adjoint bispinor $\psi(\underset{\sim}{x}, \underset{\sim}{\nu})$ is given by

$$\bar{\psi}(\underset{\sim}{x}, \underset{\sim}{\nu}) = \gamma^{o}(\psi(\underset{\sim}{x})\,\phi(\underset{\sim}{\nu})^{T})^{\dagger}\gamma^{o} = \bar{\psi}(\underset{\sim}{x}) \times (\bar{\phi}(\underset{\sim}{\nu}))^{T} \tag{47}$$

where $\bar{\psi}(\underset{\sim}{x}) = \psi(\underset{\sim}{x})^{\dagger}\gamma^{o}$ and $\bar{\phi}(\underset{\sim}{\nu}) = \phi(\underset{\sim}{\nu})^{\dagger}\gamma^{o}$ are adjoint spinors of $\psi(\underset{\sim}{x})$ and $\phi(\underset{\sim}{\nu})$ respectively.

With the expressions of $\hat{\psi}(\underset{\sim}{x})$ and $\xi(\underset{\sim}{x})$ as in (15) and (44), Eqs. (43) take the following forms:

$$\begin{aligned} \gamma^{\alpha}\partial_{\alpha}\hat{\psi}(\underset{\sim}{x}) + ie\,\gamma^{\alpha}C(\bar{\psi}(\underset{\sim}{x})\gamma_{\alpha}\psi(\underset{\sim}{x})) = 0 \\ \hat{\psi}(\underset{\sim}{x})\gamma^{\alpha^{T}}\overleftarrow{\partial}_{\alpha} + ie\,\gamma^{\alpha}C(\bar{\psi}(\underset{\sim}{x})\gamma_{\alpha}\psi(\underset{\sim}{x}) = 0 \end{aligned} \tag{48}$$

Adding these equations and using the relations

$$C^{-1}\gamma^{\mu}C = -\gamma^{\mu^{T}}$$

and

$$[\gamma^{\alpha}, \sigma^{ab}] = 2i(\eta^{\alpha a}\gamma^{b} - \eta^{\alpha b}\gamma^{a}),$$

we can arrive at the following equation:

$$\partial^{b}F_{ab}(\underset{\sim}{x}) = e(\bar{\psi}(\underset{\sim}{x})\gamma_{a}\psi(\underset{\sim}{x})) \tag{49}$$

In terms of the electromagnetic potential field $A_{\mu}(\underset{\sim}{x})$ defined in (18) and with the Lorentz condition (19), the above equation can also be written as

$$\Box A_{\mu}(\underset{\sim}{x}) = -e(\bar{\psi}(\underset{\sim}{x})\gamma_{\mu}\psi(\underset{\sim}{x})) \tag{50}$$

Equations (49) and (50) are the equations of the interacting fields in the Minkowski space-time.

Now, from Eqs. (24) and by using the following decomposition for $\psi(\underset{\sim}{x}, \underset{\sim}{\nu})$ (De, 1997):

$$\psi(\underset{\sim}{x}, \underset{\sim}{\nu}) = \psi_1(\underset{\sim}{x}) \times \phi^T(\underset{\sim}{\nu}) + \psi_2(\underset{\sim}{x}) \times \phi^{C^T}(\underset{\sim}{\nu}) \tag{51}$$

where the spinors $\psi_1(\underset{\sim}{x})$ and $\psi_2(\underset{\sim}{x})$ are eigenstates of γ^o with eigenvalues $+1$ and -1 respectively, and $\phi(\underset{\sim}{\nu})$ and $\phi^C(\underset{\sim}{\nu})$ satisfy, respectively, the equations

$$\begin{aligned} i\hbar\gamma^\mu\gamma^\ell_{h\mu}\nu^h\partial'_\ell\phi(\underset{\sim}{\nu}) &= \left(Mc - \frac{3i\hbar b_o}{2}\right)\phi(\underset{\sim}{\nu}) \\ i\hbar\gamma^\mu\gamma^\ell_{h\mu}\nu^h\partial'_\ell\phi^C(\underset{\sim}{\nu}) &= \left(Mc + \frac{3i\hbar b_o}{2}\right)\phi^C(\underset{\sim}{\nu}), \end{aligned} \tag{52}$$

it is easy to see that the bispinor satisfies the equation

$$\begin{aligned} &\left(i\hbar\gamma^\mu\partial_\mu - \frac{e}{c}\gamma^\mu\bar{A}_\mu(\underset{\sim}{x}) + \frac{3i\hbar b_o}{2}\zeta(t)\gamma^o\right)\psi(\underset{\sim}{x}, \underset{\sim}{\nu}) \\ &\quad = \frac{c}{e(t)}(m + M\zeta(t)e(t))\psi(\underset{\sim}{x}, \underset{\sim}{\nu}) \end{aligned} \tag{53}$$

To arrive at Eq. (53) we have used the relations (41) and (42) with

$$g(\underset{\sim}{x}) = \frac{1}{e^2(t)} = \exp(\pm b_o x^o) \text{ or } (b_o x^o)^n \tag{54}$$

and

$$b_o c\zeta(t) = -\frac{e'(t)}{e(t)} \tag{55}$$

Also, for the Finsler space we are considering, the connection coefficients are independent of the directional arguments of this space, and they are given by

$$\gamma^\ell_{h\mu}(\underset{\sim}{x}, \underset{\sim}{\nu}) = \zeta(t)\gamma^\ell_{h\mu} \tag{56}$$

Here, the additional mass term M appears as the constant in the process of separation of Eq. (24) and this can be considered as a manifestation of the anisotropic Finslerian character of the microscopic space-time. Equation (53) is the field equation of the associated Riemannian space of the Finsler space. This space is conformal to the Minkowski space-time. The Dirac equation for the Robertson–Walker space-time can be obtained by a pure-time transformation $\frac{dt}{e(t)} = dT$ where T is the cosmological time. We can, of course, find the field equation of the associated Riemannian space for the interacting spinor field $\psi(\underset{\sim}{x})$ through an "averaging procedure" such as

$$\psi(\underset{\sim}{x}) = \int \psi(\underset{\sim}{x}, \underset{\sim}{\nu})\chi(\underset{\sim}{\nu})\, d^4\nu \tag{57}$$

where $\chi(\underset{\sim}{\nu})$ is a (spinor) probability density or a weight function (De, 1997). One can easily find that $\psi(\underset{\sim}{x})$ satisfies the following equation:

$$\gamma^{\mu}\left(i\hbar\partial_{\mu}-\frac{e}{c}\bar{A}_{\mu}(\underset{\sim}{x})+\frac{3i\hbar b_{o}}{2}\zeta(t)\gamma^{o}\right)\psi(\underset{\sim}{x})=\frac{c}{e(t)}(m+M\zeta(t)\,e(t))\,\psi(\underset{\sim}{x}) \quad (58)$$

The field equation in local inertial frame (the flat Minkowski space-time) can be recovered from (53) by using the vierbeins

$$V_{\mu}^{\alpha}(\underset{\sim}{X})=\left(\frac{\partial y_{x}^{\alpha}}{\partial x^{\alpha}}\right)_{\underset{\sim}{x}=\underset{\sim}{X}} \qquad (\alpha=0,1,2,3)$$

which connect the curved space-time with the flat one in normal coordinates y_{x}^{α} (with index α referring to the local inertial frame) at the point $\underset{\sim}{X}$. The index μ is associated with the conformal Minkowski space-time. For fixed y_{x}^{α}, the effect of changing x^{μ} is given by $V_{\mu}^{\alpha}\rightarrow\frac{\partial x^{\nu}}{\partial x'^{\mu}}V_{\nu}^{\alpha}$. On the other hand, y_{x}^{α} may be changed by Lorentz transformation $\Lambda_{\beta}^{\alpha}(\underset{\sim}{X})$, and in this case the vierbeins $V_{\mu}^{\alpha}(\underset{\sim}{X})$ are changed to $\Lambda_{\beta}^{\alpha}(\underset{\sim}{X})V_{\mu}^{\beta}(\underset{\sim}{X})$ keeping the metric of the curved space-time invariant. Neglecting the extremely small terms we can have the field equation as

$$\gamma^{\mu}\left(i\hbar\partial_{\mu}-\frac{e}{c}A_{\mu}(\underset{\sim}{x})\right)\psi(\underset{\sim}{x})=c(m+M\zeta(t)\,e(t))\,\psi(\underset{\sim}{x}) \quad (59)$$

where $A_{\mu}(\underset{\sim}{x})=e(t)\bar{A}_{\mu}(\underset{\sim}{x})$ (expressed in local coordinates, the normal coordinates). Of course, $A_{\mu}(\underset{\sim}{x})\simeq\bar{A}_{\mu}(\underset{\sim}{x})$ as $e(t)\simeq 1$ in the present epoch of the universe. This is evident if the function $e(t)$ is expressed in terms of the cosmological time T. Equation (59) is the usual field equation in the Minkowski space-time because the additional time-dependent mass term $Mc\zeta(t)\,e(t)$ can be neglected in the present era of the universe. On the other hand, this epoch-dependent mass of the particle is dominant only in the very early universe and has cosmological consequences (De, 1993, 1999). It is to be noted that for some species of particles the epoch-dependent mass may be zero.

4. DIRECTIONAL VARIABLE-DEPENDENT FIELDS

The field $\tilde{C}(\underset{\sim}{x},\underset{\sim}{\nu})$ has been specified in (31) and (44). It is given by

$$\tilde{C}(\underset{\sim}{x},\underset{\sim}{\nu})=\gamma^{\alpha}C(\bar{\psi}(\underset{\sim}{x})\gamma_{\alpha}\psi(\underset{\sim}{x}))\times\hat{\phi}(\underset{\sim}{\nu}) \quad (60)$$

where $\hat{\phi}(\underset{\sim}{\nu})$ satisfies Eqs. (34). These equations transform into the following equation for $\hat{\phi}(\underset{\sim}{\nu})$ for its each index when we take account of the vierbein fields and connection coefficients as given in (41), (42), (54), and (56):

$$i\hbar\gamma^{\mu}\gamma_{h\mu}^{\ell}\nu^{h}\partial_{\ell}'\hat{\phi}(\underset{\sim}{\nu})=0 \quad (61)$$

In fact, for the Finsler space which we are considering, the above equation becomes

$$i\hbar \sum_{\ell=1}^{3} \gamma^{\ell} \left(\nu^{\ell} \frac{\partial}{\partial \nu^{o}} + \nu^{o} \frac{\partial}{\partial \nu^{\ell}} \right) \hat{\phi}(\underset{\sim}{\nu}) = 0 \tag{62}$$

The general form of the solution of this equation, which is homogeneous of degree zero in $\underset{\sim}{\nu}$, is given as

$$\hat{\phi}(\underset{\sim}{\nu}) = \left\{ f_1 \left(\frac{\vec{\nu}^2}{\nu^{o^2}} \right) + \frac{i\ \vec{\gamma} \cdot \vec{\nu}}{\nu^{o}} f_2 \left(\frac{\vec{\nu}^2}{\nu^{o^2}} \right) \right\} \bar{\omega} \tag{63}$$

where $\bar{\omega}$ is an arbitrary 4×4 matrix independent of $\underset{\sim}{\nu}$. Since $\hat{\phi}(\underset{\sim}{\nu})$ is symmetric, we can choose $\bar{\omega}$ to be γ^2. Here, f_1 and f_2 are functions of the homogeneous variable (of degree zero in $\underset{\sim}{\nu}$) $x = \vec{\nu}^2/\nu^{o^2}$. These functions satisfy the following equations:

$$\begin{aligned} &(1-x)\, f_1'(x) = 0 \\ &(x-3)\, f_2(x) + 2x(x-1)\, f_2'(x) = 0 \end{aligned} \tag{64}$$

The finite (for all values of $\underset{\sim}{\nu}$) and nontrivial solutions of these equations are

$$\begin{aligned} f_1(x) &= A, \\ f_2(x) &= C \frac{x-1}{x^{3/2}} H(x-1) \end{aligned} \tag{65}$$

where A and C are arbitrary constants, and

$$\begin{aligned} H(\xi) &= 1 \quad \text{for } \xi \geq 0 \\ &= 0 \quad \text{for } \xi < 0 \end{aligned}$$

Choosing $A = C = 1$, we have

$$\hat{\phi}(\underset{\sim}{\nu}) = \hat{F}(\nu^{o}, \vec{\nu})\gamma^2 \tag{66}$$

where

$$\hat{F}(\nu^{o}, \vec{\nu}) = 1 + \frac{i\ \vec{\gamma} \cdot \vec{\nu}}{\sqrt{\vec{\nu}^2}} \left(\frac{\vec{\nu}^2 - \nu^{o^2}}{\vec{\nu}^2} \right) H \left(\frac{\vec{\nu}^2 - \nu^{o^2}}{\vec{\nu}^2} \right) \tag{67}$$

It should be noted that the separation of Finslerian (lepton) field $\psi(\underset{\sim}{x}, \underset{\sim}{\nu})$ as in (46) corresponds to the case of flat associated space (by taking $e(t) \to 1$) of the Finsler space, and in this case $\phi(\underset{\sim}{\nu})$ satisfies the equation

$$i\hbar\gamma^{\mu}\gamma^{\ell}_{h\mu}\nu^{h}\partial'_{\ell}\phi(\underset{\sim}{\nu}) = Mc\phi(\underset{\sim}{\nu}) \tag{68}$$

which reduces to Eq. (61) for the case $M = 0$. On the other hand, $\psi(\underset{\sim}{x})$ satisfies Eqs. (59) and (50). For the case $M = 0$, the solution for $\phi(\underset{\sim}{\nu})$ is given by

$$\phi(\underset{\sim}{\nu}) = \hat{F}(\nu^{o}, \vec{\nu})\hat{\omega} \tag{69}$$

where $\overset{\wedge}{\omega}$ is an arbitrary spinor independent of $\underset{\sim}{\nu}$.
With the following four linearly independent spinors $\omega^b (b = 1, 2, 3, 4)$ given by

$$\omega^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \omega^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{70}$$

we can have four solutions $\phi^b(\underset{\sim}{\nu})(b = 1, 2, 3, 4)$ given as

$$\phi^b(\underset{\sim}{\nu}) = \hat{F}(\nu^o, \vec{\nu})\omega^b \tag{71}$$

Now, with the definition of adjoint bispinor, given in (45), and the separations of $\psi(\underset{\sim}{x}, \underset{\sim}{\nu})$ and its adjoint $\overline{\psi}(\underset{\sim}{x}, \underset{\sim}{\nu})$ as in (46) and (47), which are, in the present case,

$$\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu}) = \psi(\underset{\sim}{x})\,(\phi^b(\underset{\sim}{\nu}))^T,$$
$$\overline{\psi}^b(\underset{\sim}{x}, \underset{\sim}{\nu}) = (\overline{\phi}^b(\underset{\sim}{\nu}))^T\,\overline{\psi}(\underset{\sim}{x}),$$

we have for the bispinor (lepton) current

$$\overline{\psi}^b(\underset{\sim}{x}, \underset{\sim}{\nu})\gamma_\mu(\underset{\sim}{x})\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu}) = (\hat{F}(\nu^o, \vec{\nu})\omega^b\omega^{b^T}\,\hat{F}(\nu^o, \vec{\nu})\gamma^o)^T(\overline{\psi}(\underset{\sim}{x})\gamma_\mu(\underset{\sim}{x})\psi(\underset{\sim}{x}))$$

Since $\omega^b\omega^{b^T} = 1$, and $\gamma^2\hat{F}{}^T = \hat{F}\gamma^2$, we can have

$$\gamma^2\gamma^o\overline{\psi}^b(\underset{\sim}{x}, \underset{\sim}{\nu})\gamma_\mu(\underset{\sim}{x})\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu}) = \{\hat{F}(\nu^o, \vec{\nu})\}^2\gamma^2(\overline{\psi}(\underset{\sim}{x})\gamma_\mu(\underset{\sim}{x})\psi(\underset{\sim}{x}))$$

Again, noting that $q(\underset{\sim}{\nu})\hat{F}{}^{-1}\gamma^o = \gamma^o\hat{F}$ where

$$q(\underset{\sim}{\nu}) = 1 - \left(\frac{\vec{\nu}^2 - \nu^{o^2}}{\vec{\nu}^2}\right)^2 H\left(\frac{\vec{\nu}^2 - \nu^{o^2}}{\vec{\nu}^2}\right),$$

we can arrive at the following relation:

$$\overset{\wedge}{\phi}(\underset{\sim}{\nu})(\overline{\psi}(\underset{\sim}{x})\gamma_\mu(\underset{\sim}{x})\psi(\underset{\sim}{x})) = -\frac{1}{q(\underset{\sim}{\nu})}\gamma^o\overset{\wedge}{\phi}(\underset{\sim}{\nu})\overline{\psi}^b(\underset{\sim}{x}, \underset{\sim}{\nu})\gamma_\mu(\underset{\sim}{x})\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu}) \tag{72}$$

By using the vierbein fields $V_\mu^\alpha(\underset{\sim}{x})$ and $V_\alpha^\mu(\underset{\sim}{x})$ it follows from (60) that

$$\begin{aligned} \tilde{C}(\underset{\sim}{x}, \underset{\sim}{\nu}) &= \gamma^\mu(\underset{\sim}{x})C(\overline{\psi}(\underset{\sim}{x})\gamma_\mu(\underset{\sim}{x})\psi(\underset{\sim}{x})) \times \overset{\wedge}{\phi}(\underset{\sim}{\nu}) \\ &= \gamma^\mu(\underset{\sim}{x})C \times \left\{-\frac{1}{q(\underset{\sim}{\nu})}\gamma^o\overset{\wedge}{\phi}(\underset{\sim}{\nu})\overline{\psi}^b(\underset{\sim}{x}, \underset{\sim}{\nu})\gamma_\mu(\underset{\sim}{x})\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu})\right\} \end{aligned} \tag{73}$$

Similar relation also holds good for the case of nonzero M. Thus, $\tilde{C}(\underset{\sim}{x}, \underset{\sim}{\nu})$ is connected with the bispinor (lepton) current with which the electromagnetic field interacts.

The bispinor (lepton) current, as defined above, is a 4×4 matrix. Instead of it one can define a current $j_\mu(\underset{\sim}{x}, \underset{\sim}{\nu})$ as

$$j_\mu(\underset{\sim}{x}, \underset{\sim}{\nu}) = \mathrm{Tr}(\gamma^o \overline{\psi}^b(\underset{\sim}{x}, \underset{\sim}{\nu})\gamma_\mu(\underset{\sim}{x})\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu})) \tag{74}$$

We can relate this current with the "$\underset{\sim}{x}$-space" lepton current. In fact, we find that

$$j_\mu(\underset{\sim}{x}, \underset{\sim}{\nu}) = P(\underset{\sim}{\nu})(\overline{\psi}(\underset{\sim}{x})\gamma_\mu(\underset{\sim}{x})\psi(\underset{\sim}{x})) \tag{75}$$

where

$$P(\underset{\sim}{\nu}) = 4c\left[1 + \left(\frac{\vec{\nu}^2 - \nu^{o^2}}{\vec{\nu}^2}\right)^2 H\left(\frac{\vec{\nu}^2 - \nu^{o^2}}{\vec{\nu}^2}\right)\right] \tag{76}$$

With this current we have

$$\tilde{C}(\underset{\sim}{x}, \underset{\sim}{\nu}) = \gamma^\mu(\underset{\sim}{x})C \times \frac{1}{P(\underset{\sim}{\nu})} j_\mu(\underset{\sim}{x}, \underset{\sim}{\nu})\hat{\phi}(\underset{\sim}{\nu}) \tag{77}$$

Another definition for the current can also be made if we keep in mind the averaging procedure (57) for the bispinor. This current $J_\mu(\underset{\sim}{x}, \underset{\sim}{\nu})$ is given by

$$\begin{aligned} J_\mu(\underset{\sim}{x}, \underset{\sim}{\nu}) &= c\chi^\dagger(\underset{\sim}{\nu})\psi^{b^\dagger}(\underset{\sim}{x}, \underset{\sim}{\nu})\gamma^o(\underset{\sim}{x})\gamma_\mu(\underset{\sim}{x})\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu})\chi(\underset{\sim}{\nu}) \\ &= \{c\chi^\dagger(\underset{\sim}{\nu})\hat{F}^T(\nu^o, \vec{\nu})\hat{F}^T(\nu^o\ \vec{\nu})\chi(\underset{\sim}{\nu})\}(\overline{\psi}(\underset{\sim}{x})\gamma_\mu(\underset{\sim}{x})\psi(\underset{\sim}{x})) \end{aligned} \tag{78}$$

It is to be noted that for all these definitions of the current the usual continuity equation is satisfied. With this current $J_\mu(\underset{\sim}{x}, \underset{\sim}{\nu})$ we have

$$\tilde{C}(\underset{\sim}{x}, \underset{\sim}{\nu}) = \gamma^\mu(\underset{\sim}{x})C \times \frac{1}{\zeta(\underset{\sim}{\nu})} J_\mu(\underset{\sim}{x}, \underset{\sim}{\nu})\hat{\phi}(\underset{\sim}{\nu}) \tag{79}$$

where

$$\zeta(\underset{\sim}{\nu}) = c\eta^\dagger(\underset{\sim}{\nu})\eta(\underset{\sim}{\nu}) \tag{80}$$

with

$$\eta(\underset{\sim}{\nu}) = \hat{F}^T(\nu^o, \vec{\nu})\chi(\underset{\sim}{\nu}) \tag{81}$$

Since $\eta^*(\underset{\sim}{\nu}) = \hat{F}^\dagger(\nu^o, \vec{\nu})\chi^*(\underset{\sim}{\nu}) = \hat{F}(\nu^o, \vec{\nu})\chi^*(\underset{\sim}{\nu})$, $\eta^*(\underset{\sim}{\nu})$ is a solution of (61) if $\chi(\underset{\sim}{\nu})$ is independent of $\underset{\sim}{\nu}$.

We have seen that for the decomposition (51) for the bispinor $\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu})$ the field equation for it is given by (53). The continuity equation can be found to be

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\ \vec{j}\ + 3cb_o\zeta(t)\rho = 0 \tag{82}$$

where $(j^\mu) = (c\rho, \vec{j}) \equiv (j^o, \vec{j})$ is defined as above. Particularly, the definition (78) is useful here. The spinor $\chi(\underset{\sim}{\nu})$ makes the bispinor field $\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu})$ a spinor field $\phi^b(\underset{\sim}{x}, \underset{\sim}{\nu})$ which is given by

$$\phi^b(\underset{\sim}{x}, \underset{\sim}{\nu}) = \psi^b(\underset{\sim}{x}, \underset{\sim}{\nu})\chi(\underset{\sim}{\nu}) \tag{83}$$

The usual "$\underset{\sim}{x}$-space" field $\psi^b(\underset{\sim}{x})$ is obtained through the averaging procedure

$$\psi^b(\underset{\sim}{x}) = \int \phi^b(\underset{\sim}{x}, \underset{\sim}{\nu})\, d^4\nu \tag{84}$$

From (51) it follows that

$$\phi^b(\underset{\sim}{x}, \underset{\sim}{\nu}) = \psi_1(\underset{\sim}{x})\phi_1^b(\underset{\sim}{\nu}) + \psi_2(\underset{\sim}{x})\phi_2^b(\underset{\sim}{\nu}) \tag{85}$$

where

$$\begin{aligned} \phi_1^b(\underset{\sim}{\nu}) &= \phi^{b^T}(\underset{\sim}{\nu})\chi(\underset{\sim}{\nu}) \\ \phi_2^b(\underset{\sim}{\nu}) &= \phi^{c^{b^T}}(\underset{\sim}{\nu})\chi(\underset{\sim}{\nu}) \end{aligned} \tag{86}$$

Here, $\phi_1^b(\underset{\sim}{\nu})$ and $\phi_2^b(\underset{\sim}{\nu})$ are functions (not matrices) of the directional variable $\underset{\sim}{\nu}$. The spinor $\phi^b(\underset{\sim}{x}, \underset{\sim}{\nu})$ satisfies the field equation (53) for the curved "$\underset{\sim}{x}$-space." The variable $\underset{\sim}{\nu}$ appears in the field as a parameter only. Also, since $\psi_1(\underset{\sim}{x})$ and $\psi_2(\underset{\sim}{x})$ are eigenstates of γ^o with eigenvalues $+1$ and -1 respectively, we can write

$$\begin{aligned} \psi_1(\underset{\sim}{x}) &= \frac{1}{2\phi_1^b(\underset{\sim}{\nu})}(1 + \gamma^o)\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu}) \\ \psi_2(\underset{\sim}{x}) &= \frac{1}{2\phi_2^b(\underset{\sim}{\nu})}(1 - \gamma^o)\psi^b(\underset{\sim}{x}, \underset{\sim}{\nu}) \end{aligned} \tag{87}$$

Now, for the current (78) we have

$$J^\mu(\underset{\sim}{x}, \underset{\sim}{\nu}) = c\phi^{b^\dagger}(\underset{\sim}{x}, \underset{\sim}{\nu})\gamma^o(\underset{\sim}{x})\gamma^\mu(\underset{\sim}{x})\phi^b(\underset{\sim}{x}, \underset{\sim}{\nu}) \tag{88}$$

ρ and $J^k (k = 1, 2, 3)$ are given by

$$\begin{aligned} \rho &= \ell(\underset{\sim}{\nu})\psi_1^\dagger(\underset{\sim}{x})\psi_1(\underset{\sim}{x}) + m(\underset{\sim}{\nu})\psi_2^\dagger(\underset{\sim}{x})\psi_2(\underset{\sim}{x}) \\ J^k &= c\{n(\underset{\sim}{\nu})\psi_1^\dagger(\underset{\sim}{x})\gamma^o(\underset{\sim}{x})\gamma^k(\underset{\sim}{x})\psi_2(\underset{\sim}{x}) + n^*(\underset{\sim}{\nu})\psi_2^\dagger(\underset{\sim}{x})\gamma^o(\underset{\sim}{x})\gamma^k(\underset{\sim}{x})\psi_1(\underset{\sim}{x})\} \end{aligned} \tag{89}$$

where

$$\begin{aligned} \ell(\underset{\sim}{\nu}) &= \phi_1^{b*}(\underset{\sim}{\nu})\phi_1^b(\underset{\sim}{\nu}) \\ m(\underset{\sim}{\nu}) &= \phi_2^{b*}(\underset{\sim}{\nu})\phi_2^b(\underset{\sim}{\nu}) \\ n(\underset{\sim}{\nu}) &= \phi_1^{b*}(\underset{\sim}{\nu})\phi_2^b(\underset{\sim}{\nu}) \end{aligned} \tag{90}$$

It is interesting to note that the continuity equation (82) for the curved "$\underset{\sim}{x}$-space" can be written in the usual flat space form of it, that is,

$$\frac{\partial \hat{\rho}}{\partial t} + \text{div}\,\vec{\hat{j}} = 0 \tag{91}$$

where

$$\hat{j}^{\mu} = j^{\mu} \exp\left\{3cb_o \int \zeta(t)\,dt\right\} = \{e(t)\}^{-3} j^{\mu}. \text{ Therefore we have}$$

$$\hat{\rho} = \{e(t)\}^{-3}\rho \tag{92}$$

Although, $\int \hat{\rho}\,dV =$ constant in respect of the conformal time, but $\int \rho\,dV$ is not a constant. In fact, it is proportional to $\{e(t)\}^3$.

5. CONCLUDING REMARKS

In this article we have introduced the electromagnetic field, and its interaction with the lepton fields in the Finslerian microlocal space-time of extended hadrons. The quantum field equations have been obtained here from the equations of the classical fields through quantum generalization of the space-time below a fundamental length-scale. On the other hand, the classical field equations are the outcome of a property of these fields on the neighboring points of the autoparallel curve of the Finsler space. From the quantum field equations it is possible to resurrect the usual field equations of the macroscopic spaces which may either be the large-scale space-time of the universe or the laboratory space-time (the Minkowski space-time) appearing as the associated spaces of the Finsler space. This is possible for the case of the Finslerian fields separable as the direct products of the $\underset{\sim}{x}$-dependent fields with the $\underset{\sim}{\nu}$-dependent fields. The directional variable-dependent fields satisfy different equations which are solved here. The lepton current and the continuity equation in the Finsler space have also been considered. In the field equations of the macrospaces we have an additional mass of the particle, which is cosmic time-dependent. Of course, for some species of particles these additional mass terms may not arise. In fact, the constituents of hadrons do not have such mass terms. On the other hand, the $\underset{\sim}{\nu}$-dependent fields can give rise to additional quantum numbers for them for the generation of the internal symmetry of hadrons (De, 1997).

It can be seen from the classical field equations (1), (21), (22), and (26) together with the relation (2) that these equations might be expressible with the covariant $\frac{\delta}{\delta x^{\mu}} = \partial_{\mu} - N^{\ell}_{\mu}\partial'_{\ell}$ where $N^{\mu}_{\nu} = \frac{1}{2}\frac{\partial}{\partial \nu^{\nu}}(\gamma^{\mu}_{\alpha\beta}\nu^{\alpha}\nu^{\beta})$ being the nonlinear connection (in local representation). Thus, the classical fields behave like scalars under general coordinate transformation given by $x'^{\mu} = x'^{\mu}(x^{\nu})$ and $\nu'^{\mu} = X^{*\mu}_{\nu}\nu^{\nu}$ with

$X^{*\mu}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$. On the other hand, by quantum generalization the classical fields change into bispinor, and we arrive at the quantum field equations. The classical fields in the transformed coordinate system transform in the similar manner into the quantum bispinor fields and we have their corresponding field equations. Since the classical fields are scalar under this transformation there must be a mapping between the bispinors of these coordinate systems. We may imagine this mapping of the bispinor $\psi_{\alpha\beta}(\underset{\sim}{x}, \underset{\sim}{y})$ into the bispinor $\psi'_{\alpha\beta}(\underset{\sim}{x}', \underset{\sim}{y}')$ via the mappings:

(i) the bispinor $\psi_{\alpha\beta}(\underset{\sim}{x}, \underset{\sim}{y})$ into the classical field $\psi(\underset{\sim}{x}, \underset{\sim}{y})$ through "classicalization" of the space-time, and
(ii) $\psi(\underset{\sim}{x}, \underset{\sim}{y}) = \psi'(\underset{\sim}{x}', \underset{\sim}{y}')$ into $\psi'_{\alpha\beta}(\underset{\sim}{x}, \underset{\sim}{y})$ through quantum generalization of the space-time.

Thus, under the general coordinate transformations the form-invariance of a quantum field equation in the Finsler space can be understood in the following manner:

The classical fields are scalar under general coordinate transformations and their equations are form-invariant. By quantum generalization of the space-time and the corresponding mappings of the classical fields onto the bispinor fields one arrives at the corresponding form-invariant field equations. The bispinors are related by the mappings (i) and (ii) as described above. Of course, the bispinor field can behave like a scalar under a coordinate transformation. This can occur if the coordinate transformation is linear.

It is possible that the field in the coordinate space x^{μ} (the associated Riemann space) is a scalar under general coordinate transformations. In fact, the field of the coordinate space is the "$\underset{\sim}{x}$-part" of the separable bispinor $\psi_{\alpha\beta}(\underset{\sim}{x}, \underset{\sim}{y})$. On the other hand, if the bispinor is not separable as described earlier then the field in the $\underset{\sim}{x}$-space can be defined through the averaging procedure (57). It is to be noted that the bispinors are not, in general, separable in all transformed spaces, that is, the bispinors obtained under general coordinate transformations may not be separable. For the Finsler space of extended hadrons, that we have considered, the bispinors in that space are, however, separable. This is due to the fact that we are considering here a special coordinate system for the space-time of the anisotropic Finslerian microdomain. Even in measurement dynamics in flat space-time and consistent with ordinary quantum theory needs a special Lorentz frame. The necessity of such a special frame of reference was suggested by Hardy (1992) in order to block the causal paradoxes. The frame of reference in which the cosmic background radiation is isotropic is proposed there as the special one for this purpose. Percival (2000) has also proposed the need for a preferred frame like the Robertson–Walker comoving coordinate system with the standard cosmic time in order to have a consistent theory globally. Thus, in our consideration the special coordinate system for the Finslerian microscopic space-time might serve the needed one because the Robertson–Walker space-time appears here as an associated space of that space.

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